Eur. Phys. J. C 53, 133–138 (2008)
DOI 10.1140/epjc/s10052-007-0439-8

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Induced inflation from a 5D purely kinetic scalar field formalism on warped product spaces

J.E. Madriz Aguilar[a]

Departamento de Física, Universidade Federal da Paraíba, C.P. 5008-CEP: 58059-970, João Pessoa, PB 58059-970, Brazil

Received: 8 June 2007 / Revised version: 23 August 2007 /
Published online: 1 November 2007 –

Abstract. Considering a separable and purely kinetic 5D scalar field we investigate the induction of 4D scalar potentials on a 4D constant foliation on the class of 5D warped product space-times. We obtain a quantum confinement of the inflaton modes given naturally from the model for at least a class of warping factors. We can recover a 4D inflationary scenario where the inflationary potential is geometrically induced from 5D and the effective equation of state in 4D that includes the effect of the inflaton field and the induced matter is $P_{\rm eff} \simeq -\rho_{\rm eff}$.

PACS. 04.20.Jb; 11.10.kk; 98.80.Cq

1 Introduction

Warped geometries are the natural geometric background of Randall–Sundrum scenarios [1]. In braneworld scenarios this kind of geometries has been subject of great interest. Mathematically, given (M^a,γ) and (M^b,h), two Riemannian (or pseudo-Riemannian) manifolds of dimensions a and b and metrics γ and h, respectively, it is possible to construct a new Riemannian (or pseudo-Riemannian) manifold by $(M = M^a \times M^b, g)$, where by definition $g = \mathrm{e}^{2A}\gamma \otimes h$, $A : M^b \to \mathbb{R}$ being a given smooth function known as the warping factor [2]. An interesting property of this kind of geometries is the natural splitting that naturally occurs between the dynamics in the 4D space-time and the dynamics in the extra dimensions. In the particular case of 5D spaces we can separate the motion in the fifth coordinate from the one in the remaining 4D usual space-time dimensions. This feature could be of great "wealth" in some of the new treatments of inflation in theories in more than four dimensions.

Some of the most popular theories in higher dimensions are the Kaluza–Klein theory [3–6], where the fifth dimension is compactified, the induced matter theory or in more general terms noncompactified $(n+1)$ general relativity [7–9] and the braneworld scenarios inspired in string theory known as M-theory [10–15]. The universe in braneworld cosmologies is considered as a brane embedded in a higher dimensional space-time called the bulk. In these models only gravity and some exotic matter like dilatonic scalar fields can propagate through the bulk, while our observable universe is confined to a particular brane hypersurface [16]. Several mechanisms of confinement of matter to a special $(3+1)$ hypersurface have been implemented. The use of a confining potential is one of these mechanisms [17]. However, this continues to be a central issue not only in braneworld scenarios but also in the induced matter theory.

New approaches to inflationary cosmology based on a scalar field formalism have been proposed in the context of higher dimensional theories of gravity. In particular, using ideas of the induced matter theory a new formalism for describing inflation [18–20], scalar metric fluctuations and gravitational waves [21–23] has recently been introduced. Some other topics, as for example effects of a decaying cosmological parameter in this recent 5D framework, were also studied [24]. The basic idea of this formalism is the existence of a 5D space-time equipped with a purely kinetic scalar field φ, in which our observable 4D universe can be confined to a particular hypersurface. The induced 4D inflationary dynamics is described in terms of the induced 4D scalar field $\varphi(x,\psi_0)$, being ψ_0 the value of the fifth coordinate specified by a given foliation of the 5D space-time, where the particular choice of the hypersurface $\psi = H^{-1}$, with H the Hubble parameter, is crucial for its physical predictions.

In this letter we develop a new approach in a more general manner. We start considering a purely kinetic 5D scalar field on a warped product geometrical background. Assuming the separability of the scalar field and assuming that the 5D space-time can be foliated, we present a general mechanism for inducing a scalar potential of a true 4D scalar field $\phi(x)$ instead of a potential for a 4D effective scalar field $\varphi(x,\psi_0)$. As an application of the formalism we study de Sitter inflationary cosmology recovering the usual 4D formalism but with a geometrically induced 4D scalar potential $V(\phi)$. As we will see, the quantum confinement

[a] e-mail: jemadriz@fisica.ufpb.br

of the inflaton modes is obtained naturally from the model, and physical predictions can be made independent of the hypersurface chosen.

2 General formalism

We consider a 5D space-time (M, g_{AB}), where g_{AB} is a 5D metric which in the local coordinates (y^A) is expressed by the line element $\mathrm{d}S^2 = g_{AB}\,\mathrm{d}y^A\mathrm{d}y^B$. The action that describes a purely kinetic 5D scalar field $\varphi(y^A)$ minimally coupled to gravity can be written as

$$^{(5)}\mathcal{S} = \int \mathrm{d}^5y\sqrt{|g|}\left[\frac{^{(5)}\mathcal{R}}{2\kappa^2} + {}^{(5)}\mathcal{L}(\varphi,\varphi_{,A})\right], \tag{1}$$

where $^{(5)}\mathcal{R}$ is the scalar curvature, κ defines the 5D gravitational coupling and the Lagrangian density for the scalar field is

$$^{(5)}L(\varphi,\varphi_{,A}) = \sqrt{|g|}\,{}^{(5)}\mathcal{L}(\varphi,\varphi_{,A}) = \sqrt{|g|}\left[\frac{1}{2}g^{AB}\varphi_{,A}\varphi_{,B}\right]. \tag{2}$$

The variation of the action with respect to the metric and the scalar field, respectively, gives the 5D field equations

$$G_{AB} = \kappa T_{AB}\,, \tag{3}$$

$$^{(5)}\Box\varphi \equiv \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial y^A}\left[\sqrt{|g|}g^{AB}\varphi_{,B}\right] = 0\,, \tag{4}$$

$G_{AB} = R_{AB} - (1/2){}^{(5)}\mathcal{R}g_{AB}$ being the Einstein tensor in 5D and T_{AB} the energy-momentum tensor for a free scalar field specified by

$$T_{AB} = \varphi_{,A}\varphi_{,B} - \frac{1}{2}g_{AB}\varphi^{,C}\varphi_{,C}\,. \tag{5}$$

In the local coordinates $(y^A) = (x^\mu, \psi)$, we consider the class of warped geometries given by the line element

$$\mathrm{d}S^2 = \mathrm{e}^{2A(\psi)}h_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - \mathrm{d}\psi^2\,, \tag{6}$$

where $h_{\mu\nu}$ is a 4D metric with determinant $\det(h_{\mu\nu}) = h$. On this metric background (4) reads

$$^{(4)}\Box\varphi - \mathrm{e}^{-2A(\psi)}\frac{\partial}{\partial\psi}\left[\mathrm{e}^{4A(\psi)}\frac{\partial\varphi}{\partial\psi}\right] = 0\,, \tag{7}$$

where $^{(4)}\Box\varphi = (1/\sqrt{-h})(\partial/\partial x^\mu)(\sqrt{-h}h^{\mu\nu}\varphi_{,\nu})$. Assuming a separable scalar field $\varphi(x,\psi) = \phi(x)Q(\psi)$, (7) yields the system

$$^{(4)}\Box\phi(x) - \alpha\phi(x) = 0\,, \tag{8}$$

$$\frac{\mathrm{d}^2Q}{\mathrm{d}\psi^2} + 4\frac{\mathrm{d}A(\psi)}{\mathrm{d}\psi}\frac{\mathrm{d}Q}{\mathrm{d}\psi} - \alpha\mathrm{e}^{-2A(\psi)}Q = 0\,, \tag{9}$$

with α a separation constant. This is a system of ordinary differential equations of second order which in principle, for a given warping factor $A(\psi)$, can be solved.

We can write (2) as

$$^{(5)}L(\varphi,\varphi_{,A}) = \mathrm{e}^{2A(\psi)}Q^2(\psi)\sqrt{-h} \times\left[\frac{1}{2}h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\mathrm{e}^{2A(\psi)}\left(\frac{\overset{\star}{Q}}{Q}\right)^2\phi^2(x)\right], \tag{10}$$

where the star $(\star)$ denotes $\partial/\partial\psi$. Now assuming that the 5D space-time can be foliated by a family of hypersurfaces defined by $\Sigma_0 : \psi = \psi_0$, with ψ_0 a constant, (10) evaluated on one of these hypersurfaces becomes

$$^{(4)}L(\phi,\phi_{,\mu}) \equiv {}^{(5)}L_{\Sigma_0}(\varphi,\varphi_{,A}) = \mathrm{e}^{2A(\psi_0)}Q^2(\psi_0)\sqrt{-h}\left[\frac{1}{2}h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V_{\mathrm{ind}}(\phi)\right], \tag{11}$$

where $V_{\mathrm{ind}}(\phi)$ is the induced 4D scalar potential given by

$$V_{\mathrm{ind}}(\phi) = \frac{1}{2}\mathrm{e}^{2A(\psi_0)}\left(\frac{\overset{\star}{Q}}{Q}\right)^2\Bigg|_{\psi_0}\phi^2(x)\,, \tag{12}$$

which is different for every warping factor. Clearly the induction of the potential depends strongly on the separability condition of φ and on the splitting property of the warped product metrics. Hence the action (1) evaluated on Σ_0 reads

$$^{(4)}\mathcal{S} = \int \mathrm{d}^4x\sqrt{-h}\left[\frac{^{(4)}\mathcal{R}}{16\pi G} + \mathcal{L}_{\mathrm{IM}} + \frac{1}{2}h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V_{\mathrm{ind}}(\phi)\right], \tag{13}$$

where we have made the identification $\kappa^2Q^2(\psi_0) = 8\pi G$, and $\mathcal{L}_{\mathrm{IM}}$ is a 4D induced matter Lagrangian density coming from the part of $^{(5)}\mathcal{R}$ that depends on the fifth coordinate ψ [7–9]. The action (13) is interpreted as a 4D induced action, which is constructed on Σ_0 by means of the 4D induced Lagrangian (11). This action, besides inducing matter in a geometrical manner, allows one to describe the dynamics of the 4D scalar field $\phi(x)$ minimally coupled to gravity with a geometrically induced scalar potential $V_{\mathrm{ind}}(\phi)$. The fact that we have a geometrical mechanism of deriving a scalar potential avoiding its introduction "by hand" is an attractive feature of this approach. Moreover, as we shall see in Sect. 4, the fact that we have a scalar potential induced geometrically entails a physical scalar field mass parameter also geometrical in origin, and this is a valuable characteristic specially in inflationary frameworks. Compared with standard 4D inflationary scenarios this feature represents an advantage in the sense that for example it can be possible to treat de Sitter inflation without the introduction a priori of a small mass parameter for the inflaton field. This is basically because in the kind of formalism like the present one it is possible to treat a de Sitter expansion without a constant scalar potential by simply choosing a foliation $\psi = \psi_0 = H_0^{-1}$, H_0 being the constant Hubble parameter [25, 26].

The field equations according to (13) are then

$$^{(4)}G_{\mu\nu} = 8\pi G\, ^{(4)}T^{(\phi)}_{\mu\nu} + 8\pi G\, ^{(4)}T^{(\mathrm{IM})}_{\mu\nu}\,, \quad (14)$$

$$^{(4)}\Box\phi + V'_{\mathrm{ind}}(\phi) = 0\,, \quad (15)$$

where $(')$ denotes the derivative with respect to the field ϕ and

$$^{(4)}T^{(\mathrm{IM})}_{\mu\nu} = \frac{3}{8\pi G}\left[\left(\frac{\mathrm{d}^2A}{\mathrm{d}\psi^2}\right)_{\psi=\psi_0} + 2\left(\frac{\mathrm{d}A}{\mathrm{d}\psi}\right)^2_{\psi=\psi_0}\right]\mathrm{e}^{2A(\psi_0)}h_{\mu\nu} \quad (16)$$

is the induced matter energy-momentum tensor on the geometrical background (6).

3 Quantum cosmological confinement of the 5D scalar modes

As an illustration of the previous formalism we shall study the cosmological case. In order to describe a 5D space-time with a 3D spatial expansion that in addition is spatially flat, we consider the line element

$$\mathrm{d}S^2 = \mathrm{e}^{2A(\psi)}[\mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2] - \mathrm{d}\psi^2\,, \quad (17)$$

where $a(t)$ is the scale factor, t is the cosmic time and $\mathrm{d}r^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, $\{x, y, z\}$ being the usual Cartesian coordinates. The 5D equation of motion (7) now reads

$$\ddot{\varphi} + 3H(t)\dot{\varphi} - \frac{1}{a(t)}\nabla^2_r\varphi - \mathrm{e}^{2A(\psi)}\left[4\frac{\mathrm{d}A(\psi)}{\mathrm{d}\psi}\frac{\partial\varphi}{\partial\psi} + \frac{\partial^2\varphi}{\partial\psi^2}\right] = 0\,, \quad (18)$$

$H(t) = \dot{a}/a$ being the Hubble parameter and the dot denoting the time derivative. Following a canonical quantization process we impose the equal time commutation relations

$$\left[\hat{\varphi}(t,\mathbf{r},\psi), \hat{\Pi}^t_\varphi(t,\mathbf{r}',\psi')\right] = \mathrm{i}\frac{1}{a^3}\mathrm{e}^{-2A(\psi)}\delta^{(3)}(\mathbf{r}-\mathbf{r}')\delta(\psi-\psi')\,, \quad (19)$$

$$\left[\hat{\varphi}(t,\mathbf{r},\psi), \hat{\varphi}(t,\mathbf{r}',\psi')\right] = \left[\hat{\Pi}^t_\varphi(t,\mathbf{r},\psi), \hat{\Pi}^t_\varphi(t,\mathbf{r}',\psi')\right] = 0\,, \quad (20)$$

where $\Pi^t_\varphi = \frac{\partial\, ^{(5)}L}{\partial\dot{\varphi}} = \sqrt{|^{(5)}g|}g^{tt}\dot{\varphi} = e^{2A(\psi)}a^3\dot{\varphi}$ is the momentum conjugate to φ.

The operator $\hat{\varphi}$ is decomposed in Fourier modes as follows:

$$\hat{\varphi}(t,\mathbf{r},\psi) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k_r\,\mathrm{d}k_\psi\left[\hat{a}_{k_rk_\psi}\xi_{k_rk_\psi}(t,\psi)\mathrm{e}^{\mathrm{i}\mathbf{k}_r\cdot\mathbf{r}} + \hat{a}^\dagger_{k_rk_\psi}\xi^*_{k_rk_\psi}(t,\psi)\mathrm{e}^{-\mathrm{i}\mathbf{k}_r\cdot\mathbf{r}}\right]\,, \quad (21)$$

where the annihilation and creation operators $\hat{a}_{k_rk_\psi}$ and $\hat{a}^\dagger_{k_rk_\psi}$ satisfy the canonical commutation algebra

$$\left[\hat{a}_{k_rk_\psi}, \hat{a}^\dagger_{k'_rk'_\psi}\right] = \delta^{(3)}(\mathbf{k}_r - \mathbf{k}'_\mathbf{r})\delta(k_\psi - k'_\psi)\,, \quad (22)$$

$$\left[\hat{a}_{k_rk_\psi}, \hat{a}_{k'_rk'_\psi}\right] = \left[\hat{a}^\dagger_{k_rk_\psi}, \hat{a}^\dagger_{k'_rk'_\psi}\right] = 0\,, \quad (23)$$

while the Wronskian condition for the k_rk_ψ-modes yields

$$\xi_{k_rk_\psi}\dot{\xi}^*_{k_rk_\psi} - \dot{\xi}_{k_rk_\psi}\xi^*_{k_rk_\psi} = \mathrm{i}\frac{1}{a^3}\mathrm{e}^{-2A(\psi)}\,, \quad (24)$$

with the asterisk $(*)$ denoting the complex conjugate. Inserting (21) in (18) we obtain

$$\ddot{\xi}_{k_rk_\psi} + 3H\dot{\xi}_{k_rk_\psi} + \left[a^{-2}k_r^2 - \mathrm{e}^{2A(\psi)}\left(4\overset{\star}{A}\frac{\partial}{\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right)\right]\xi_{k_rk_\psi} = 0\,, \quad (25)$$

which is the dynamical equation for the modes $\xi_{k_rk_\psi}(t,\psi)$. By introducing the auxiliary field

$$\zeta_{k_rk_\psi}(t,\psi) = \exp\left[\frac{3}{2}\int H(t)\,\mathrm{d}t\right]\exp[2A(\psi)]\xi_{k_rk_\psi}(t,\psi)\,, \quad (26)$$

we can write (25) as

$$\ddot{\zeta}_{k_rk_\psi} + \left[a^{-2}k_r^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} - \mathrm{e}^{2A(\psi)}\left(4\overset{\star}{A}{}^2 + 2\overset{\star\star}{A} + \frac{\partial^2}{\partial\psi^2}\right)\right]\zeta_{k_rk_\psi} = 0\,. \quad (27)$$

Assuming that the k_rk_ψ-modes can be separated in the form $\zeta_{k_rk_\psi}(t,\psi) = \zeta_{k_r}(t)\zeta_{k_\psi}(\psi)$, (25) yields

$$\ddot{\zeta}_{k_r} + \left[a^{-2}k_r^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} - \beta^2\right]\zeta_{k_r} = 0\,, \quad (28)$$

$$\overset{\star\star}{\zeta}_{k_\psi} + [4\overset{\star}{A}{}^2 + 2\overset{\star\star}{A} - \beta^2\mathrm{e}^{-2A(\psi)}]\zeta_{k_\psi} = 0\,, \quad (29)$$

β being a separation constant. The Wronskian condition (24) now becomes

$$\zeta_{k_r}\dot{\zeta}^*_{k_r} - \dot{\zeta}_{k_r}\zeta^*_{k_r} = \mathrm{i}, \quad \zeta_{k_\psi}\zeta^*_{k_\psi} = \mathrm{e}^{2A(\psi)}\,. \quad (30)$$

Equation (29) means that the stability of the k_ψ-modes depends strongly on the warping factor. In other words, the mode dynamics along the fifth dimension is driven by the warping factor. Moreover, as we saw in the previous section the warping factor not only controls the dynamics of the modes in the k_ψ-direction, but it also is a preponderant factor in the induction of the scalar potential $V(\phi)$ on Σ_0.

As an illustrative example we consider the warping factor $A(\psi) = \ln(\psi/\psi_0)$, which corresponds to the line element

$$\mathrm{d}S^2 = \left(\frac{\psi}{\psi_0}\right)^2\left[\mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2\right] - \mathrm{d}\psi^2\,, \quad (31)$$

which is a generalization of the Ponce de Leon metric [27] on which $a(t) = \exp(2t/\psi_0)$. Thus, (29) reduces to

$$\overset{\star\star}{\zeta}_{k_\psi} + \left(\frac{2 - \beta^2\psi_0^2}{\psi^2}\right)\zeta_{k_\psi} = 0\,, \quad (32)$$

whose general solution is given by

$$\zeta_{k_\psi}(\psi) = B_1(k_\psi)\psi^{\frac{1}{2}\left(1+\sqrt{4\beta^2\psi_0^2-7}\right)} + B_2(k_\psi)\psi^{\frac{1}{2}\left(1-\sqrt{4\beta^2\psi_0^2-7}\right)} . \quad (33)$$

Considering for simplicity a de Sitter expansion $a(t) = a_0 \mathrm{e}^{H_0 t}$, the equation of motion for the k_r-modes (28) now reads

$$\ddot{\zeta}_{k_r} + \left[k_r^2 a_0^{-2} \mathrm{e}^{-2H_0 t} - \frac{9}{4}H_0^2 - \beta^2\right]\zeta_{k_r} = 0 . \quad (34)$$

The general solution of (34) has the form

$$\zeta_{k_r}(t) = D_1(k_r)\mathcal{H}_\nu^{(1)}\left[\frac{k_r}{a_0 H_0}\mathrm{e}^{-H_0 t}\right] + D_2(k_r)\mathcal{H}_\nu^{(2)}\left[\frac{k_r}{a_0 H_0}\mathrm{e}^{-H_0 t}\right] , \quad (35)$$

where $\nu = 1/(2H_0)\sqrt{9H_0^2 + 4\beta^2}$ and $\mathcal{H}_\nu^{(1,2)}$ are the first and second kind Hankel functions. Using the normalization condition (30), which now is $\xi_{k_\psi}\xi^*_{k_\psi} = (\psi/\psi_0)^2$, and selecting the Bunch–Davies vacuum [28] for de Sitter space $B_2(k_\psi) = 0$, $D_1(k_r) = 0$, we obtain the result that the unique normalizable k_ψ-mode is found by setting $\beta = \pm\sqrt{2}/\psi_0$. In this case the normalized solution for the $k_r k_\psi$-modes $\zeta_{k_r k_\psi}$ is

$$\bar{\zeta}_{k_r}(t,\psi) \equiv \zeta_{k_r k_\psi}(t,\psi) = \frac{\mathrm{i}}{2}\sqrt{\frac{\pi}{H_0}}\left(\frac{\psi}{\psi_0}\right)\mathcal{H}_\nu^{(2)}\left[\frac{k_r}{a_0 H_0}\mathrm{e}^{-H_0 t}\right] , \quad (36)$$

and therefore

$$\bar{\xi}_{k_r}(t) \equiv \xi_{k_r k_\psi}(t,\psi) = \frac{\mathrm{i}}{2}\sqrt{\frac{\pi}{H_0}}\mathrm{e}^{-\frac{3}{2}H_0 t}\mathcal{H}_\nu^{(2)}\left[\frac{k_r}{a_0 H_0}\mathrm{e}^{-H_0 t}\right] , \quad (37)$$

with $\nu = [1/(2H_0)]\sqrt{9H_0^2 + 8\psi_0^{-2}}$. Clearly, the modes $\bar{\xi}_{k_r}$ do not exhibit any dependence of the fifth coordinate. This fact can be interpreted as a kind of confinement of the quantum scalar modes $\bar{\xi}_{k_r}(t)$ on the hypersurfaces $\Sigma_0 : \psi = \psi_0$, since the modes of φ do not propagate along the fifth dimension. Note that this confinement is obtained naturally from the theory without the introduction of a confining scalar potential.

4 4D induced inflation

Now we are able to treat the 4D inflationary case derived from the induced 4D action (13). According to the formalism exposed in Sect. 2, the induced 4D line element is given by

$$\mathrm{d}s^2 = h_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2 . \quad (38)$$

According to (14) and (16) the induced matter density and pressure can be separated as

$$\rho_{\mathrm{eff}} = \rho_\phi + \rho_{(\mathrm{IM})} , \quad P_{\mathrm{eff}} = P_\phi + P_{(\mathrm{IM})} , \quad (39)$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\nabla_r\phi)^2 + V_{\mathrm{ind}}(\phi) , \quad (40)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\nabla_r\phi)^2 - V_{\mathrm{ind}}(\phi) , \quad (41)$$

$$\rho_{(\mathrm{IM})} = \frac{3}{8\pi G}\left[\left(\frac{\mathrm{d}^2 A}{\mathrm{d}\psi^2}\right)_{\psi=\psi_0} + 2\left(\frac{\mathrm{d}A}{\mathrm{d}\psi}\right)^2_{\psi=\psi_0}\right] = -P_{(\mathrm{IM})} . \quad (42)$$

In this way we have two contributions; one due to the scalar field and another one due to the assumed action of the fifth dimension of inducing matter on the 4D space-time [7–9]. The effective equation of state for slow-roll conditions on the inflaton field ϕ can be written as $P_{\mathrm{eff}} \simeq -\rho_{\mathrm{eff}}$.

In order to determine the exact form of the induced scalar potential (12), we consider for simplicity the same warping factor $A(\psi) = \ln(\psi/\psi_0)$. Under these considerations (9) has the general solution

$$Q(\psi) = C_1\psi^{(1/2)\left[-3+\sqrt{9-4\alpha\psi_0^2}\right]} + C_2\psi^{(-1/2)\left[3+\sqrt{9-4\alpha\psi_0^2}\right]} . \quad (43)$$

Choosing $C_2 = 0$, the 4D induced scalar potential (12) reads

$$V_{\mathrm{ind}}(\phi) = \frac{1}{2}\left[\frac{\sqrt{9-4\alpha\psi_0^2}-3}{2\psi_0}\right]^2\phi^2(t,\mathbf{r}) . \quad (44)$$

The field equation for the scalar field $\phi(t,\mathbf{r})$ is then

$$\ddot{\phi} + 3H(t)\dot{\phi} - a^{-2}(t)\nabla_r^2\phi + \left(\frac{\sqrt{9-4\alpha\psi_0^2}-3}{2\psi_0}\right)^2\phi = 0 . \quad (45)$$

The field operators $\hat{\phi}(t,\mathbf{r})$ and $\hat{\Pi}^t_\phi = \partial\,{}^{(4)}L/(\partial\dot{\phi}) = a^3\dot{\phi}$ satisfy the equal time algebra

$$\left[\hat{\phi}(t,\mathbf{r}), \hat{\Pi}^t_\phi(t,\mathbf{r}')\right] = \mathrm{i}\frac{1}{a^3}\delta^{(3)}(\mathbf{r}-\mathbf{r}') ,$$
$$\left[\hat{\phi}(t,\mathbf{r}), \hat{\phi}(t,\mathbf{r}')\right] = \left[\hat{\Pi}^t_\phi(t,\mathbf{r}), \hat{\Pi}^t_\phi(t,\mathbf{r}')\right] = 0 . \quad (46)$$

Thus, the field operator $\hat{\phi}(t,\mathbf{r})$ can be decomposed in Fourier modes as follows:

$$\hat{\phi}(t,\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3 k_r\left[\hat{a}_{k_r}h_{k_r}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}_r\cdot\mathbf{r}} + \hat{a}^\dagger_{k_r}h^*_{k_r}(t)\mathrm{e}^{-\mathrm{i}\mathbf{k}_r\cdot\mathbf{r}}\right] , \quad (47)$$

where the annihilation and creation operators $\hat{a}_{k_r}$ and $\hat{a}^\dagger_{k_r}$ satisfy the commutation relations

$$\left[\hat{a}_{k_r}, \hat{a}^\dagger_{k'_r}\right] = \delta^{(3)}(\mathbf{r}-\mathbf{r}') , \quad \left[\hat{a}_{k_r}, \hat{a}_{k'_r}\right] = \left[\hat{a}^\dagger_{k_r}, \hat{a}^\dagger_{k'_r}\right] = 0 . \quad (48)$$

The Wronskian condition for the k_r-modes h_{k_r} is given by

$$h_{k_r}\dot{h}^*_{k_r} - \dot{h}_{k_r}h^*_{k_r} = \frac{\mathrm{i}}{a^3}\,. \qquad (49)$$

Now, implementing the transformation $\phi(t,\mathbf{r}) = \exp[(-3/2)\int H(t)\,\mathrm{d}t]\chi(t,\mathbf{r})$, (45) becomes

$$\ddot{\chi} - a^{-2}\nabla^2_r\chi - \left[\frac{9}{4}H^2 + \frac{3}{2}\dot{H} - m^2\right]\chi = 0\,, \qquad (50)$$

where $m^2 = [(\sqrt{9-4\alpha\psi_0^2}-3)/(2\psi_0)]^2$ plays the role of a geometrical inflaton "mass". Hence, the following Fourier expansion is valid:

$$\chi(t,\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k_r\left[\hat{a}_{k_r}u_{k_r}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}_r\cdot\mathbf{r}} + \hat{a}^\dagger_{k_r}u^*_{k_r}(t)\mathrm{e}^{-\mathrm{i}\mathbf{k}_r\cdot\mathbf{r}}\right], \qquad (51)$$

where now $u_{k_r}(t) = \exp[(3/2)\int H(t)\,\mathrm{d}t]h_{k_r}(t)$ and $u^*_{k_r}(t) = \exp[(3/2)\int H(t)\,\mathrm{d}t]h^*_{k_r}(t)$. In terms of the new k_r-modes u_{k_r}, the Wronskian condition (49) reads

$$u_{k_r}\dot{u}^*_{k_r} - \dot{u}_{k_r}u^*_{k_r} = \mathrm{i}\,. \qquad (52)$$

Considering for simplicity a de Sitter expansion $a(t) = a_0\exp[H_0t]$, the evolution equation for the k_r-modes u_{k_r} is

$$\ddot{u}_{k_r} + \left[k_r^2a_0^{-2}\mathrm{e}^{-2H_0t} - \frac{9}{4}H_0^2 - m^2\right]u_{k_r} = 0\,. \qquad (53)$$

Stable solutions of this equation can be obtained for $k_r > k_0$, where $k_0(t) = [(9/4)H_0^2 + m^2]^{1/2}\exp[H_0t]$. The general solution of (53) is given by

$$u_{k_r}(t) = F_1\mathcal{H}^{(1)}_{\nu_s}[Z(t)] + F_2\mathcal{H}^{(2)}_{\nu_s}[Z(t)]\,, \qquad (54)$$

with $\nu_s = [1/(2H_0)]\sqrt{9H_0^2+4m^2}$ and $Z(t) = [k_r/(a_0H_0)\,\mathrm{e}^{-H_0t}]$, and F_1 and F_2 being integration constants. Selecting the Bunch–Davies vacuum $F_1 = 0$ the usual normalization is achieved by using (52). Therefore, a normalized solution is

$$u_{k_r}(t) = \frac{\mathrm{i}}{2}\sqrt{\frac{\pi}{H_0}}\mathcal{H}^{(2)}_{\nu_s}[Z(t)]\,. \qquad (55)$$

The squared quantum fluctuations on the IR-sector ($k_r \ll k_0$) are given by

$$\langle\phi^2(t)\rangle_{\mathrm{IR}} = \frac{\mathrm{e}^{-3H_0t}}{2\pi^2}\int_0^{\upsilon k_H}\frac{\mathrm{d}k_r}{k_r}k_r^3[u_{k_r}(t)u^*_{k_r}(t)]\Big|_{\mathrm{IR}}\,, \qquad (56)$$

where $\upsilon = k^{\mathrm{IR}}_{\mathrm{max}}/k_{\mathrm{p}} \ll 1$ is a dimensionless parameter, and $k^{\mathrm{IR}}_{\mathrm{max}} = k_{\mathrm{H}}(t_i) = k_0(t_i)$ is the wave number related to the Hubble radius at the time t_i when the modes re-enter the horizon. Moreover, k_{p} is the Planckian wave number. For a Hubble parameter value $H_0 = 0.5\times10^{-9}M_{\mathrm{p}}$, values of υ within the interval 10^{-5} to 10^{-8} correspond to a number of e-foldings of $N_{\mathrm{e}} = 63$ [29]. Thus, considering the asymptotic expansion for the Hankel function $\mathcal{H}^{(2)}_{\nu_s}[Z(t)] \simeq (-\mathrm{i}/\pi)\Gamma(\nu_s)(Z/2)^{-\nu_s}$, (56) becomes

$$\langle\phi^2\rangle = \frac{2^{2\nu_s}}{8\pi}\frac{a_0^3H_0^2}{3-2\nu_s}\Gamma^2(\nu_s)\left(\frac{k_r}{a(t)H_0}\right)^{3-2\nu_s}\Bigg|_{\upsilon k_{\mathrm{H}}}\,, \qquad (57)$$

and the corresponding power spectrum $\mathcal{P}_\phi(k_r) \sim [k_r/(aH_0)]^{3-2\nu_s}$. Clearly, this spectrum becomes nearly scale invariant when $m \ll 1$. Furthermore, the spectral index is given by $n_s = 4-2\nu_s = 4-3\sqrt{1+[2m/(3H_0)]^2}$. Hence, knowing from the observational data that [30] $0.94 \le n_s \le 1$, we can establish that $0 \le m \le 0.3H_0$. This interval corresponds to $-1/4(H_0/\psi_0)[0.36H_0\psi_0 + 3.6] \le \alpha \le 0$. Therefore, for every hypersurface $\psi = \psi_0$, the value of α can change in such a way that m keeps on having values in the same interval.

5 Final comments and conclusions

We have studied a scalar field formalism from a pure kinetic 5D scalar field φ on the class of 5D warped product spaces. Within this new approach we have developed a consistent way to induce a 4D Lagrangian density for a true 4D scalar field ϕ, from a 5D Lagrangian density that describes the kinetic scalar field φ. As a first application of the formalism we have studied inflationary cosmology. This new approach enables us to have a 4D inflationary formalism where the scalar potential $V(\phi)$ is induced due to the motion of the field φ with respect to the fifth coordinate ψ, in an special manner. The 4D formalism is constructed on a family of hypersurfaces given by the constant foliation $\Sigma_0 : \psi = \psi_0$. Something that is very important to stress is that the formalism is based strongly on the separability of the field φ and on the properties of the warped product spaces. Indeed, the applicability of the general mechanism of induction of 4D potentials could go beyond cosmology. This is because the minimal conditions in order for the mechanism to work are the separability of the 5D scalar field φ and the separability of the components of the metric $g_{\mu\nu}(x,\psi)$ and $g_{\psi\psi}(x,\psi)$. Of course the class of warped product spaces fits in this. In the case of a space-time dependence for $g_{\psi\psi}$ the induced 4D potential will be of the form $V[x,\phi(x)] = (1/2)M^2(x)\phi^2(x)$. In other words, the induced potential not only will depend on the true scalar field ϕ, but also on the space-time coordinates. In this way the mechanism could explain in a geometrical manner with an extra dimension the appearance of local scalar potentials that have an extra space-time dependence in contexts that are not necessarily cosmological. In an inflationary scenario we have obtained the result that at least for a class of warping factors the quantum modes associated to the 5D scalar field φ exhibit a natural quantum confinement to the 4D hypersurfaces Σ_0. This characteristic is merely of origin quantum, since it is derived as a consequence of the employment of the quantum commutation relations in 5D. An interesting implication of this quantum behavior is that for instance for a warping factor $A(\psi) = \ln(\psi/\psi_0)$, if we choose $\psi_0 = \sqrt{3/\Lambda}$ with Λ the cosmological constant, 5D classical particles do not have confinement to hypersurfaces solely due to gravitational effects [2]. However, in the case of a 5D quantum scalar field confinement of the quantum modes is achieved naturally. Finally, as is well known in many higher dimensional cosmological theories, one issue consists in explaining the fact that the observable universe seems to be confined to a particular hypersurface. In our formalism the part of the 4D dynamics that in principle could depend on the value of ψ_0 is the part that contains the information of the 4D induced potential $V(\phi) = (1/2)m^2\phi^2(x)$. However, the presence of a separation constant in (9) makes it possible that m, as defined by (12), can remain invariant for any value of ψ_0. One example of this fact is the final result of the de Sitter inflationary model in this letter, where the interval $0 \leq m \leq 0.3H_0$ is maintained invariant for $-1/4(H_0/\psi_0)[0.36H_0\psi_0 + 3.6] \leq \alpha \leq 0$, independently of the value of ψ_0.

Acknowledgements. JEMA acknowledges CNPq-CLAF and UFPB (Brazil) for financial support.

References

1. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999)
2. F. Dahia, C. Romero, L.F.P. da Silva, R. Tavakol, gr-qc/0702063
3. T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. **33**, 996 (1921)
4. O. Klein, Z. Phys. **37**, 895 (1926)
5. O. Klein, Nature **118**, 516 (1926)
6. T. Appelquiat, A. Chodos, P.G.O. Freud, Modern Kaluza–Klein Theories (Addison-Wesley, Reading, MA, 1987)
7. P.S. Wesson, Space-Time-Matter Modern Kaluza–Klein Theory (World Scientific, Singapore, 1999)
8. P.S. Wesson, J. Ponce de Leon, J. Math. Phys. **33**, 3883 (1992)
9. J.M. Overduin, P.S. Wesson, Phys. Rep. **283**, 303 (1997) [gr-qc/9805018]
10. N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B **429**, 263 (1998)
11. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B **436**, 257 (1998)
12. D. Youm, Phys. Rev. D **62**, 084 002 (2000)
13. P.S. Wesson, B. Mashhoon, H. Liu, W.N. Sajko, Phys. Lett. B **456**, 34 (1999)
14. J. Ponce de Leon, Phys. Lett. B **523**, 311 (2001)
15. R. Maartens, Living Rev. Relat. **7**, 1 (2004) [gr-qc/0312059]
16. P. Brax, C. Vande Bruck, Class. Quantum Grav. **20**, R201 (2003) [hep-ph/046279]
17. M. Heydari-Fard, M. Shirazi, S. Jalalzadeh, H.R. Sepangi, Phys. Lett. B **64**, 1 (2006)
18. J.E.M. Aguilar, M. Bellini, Phys. Lett. B **596**, 116 (2004)
19. J.E.M. Aguilar, M. Bellini, Eur. Phys. J. C **38**, 367 (2004)
20. M. Anabitarte, J.E.M. Aguilar, M. Bellini, Eur. Phys. J. C **45**, 249 (2006)
21. J.E.M. Aguilar, Phys. Lett. B **645**, 6 (2007)
22. J.E.M. Aguilar, M. Anabitarte, M. Bellini, Phys. Lett. B **632**, 6 (2006)
23. M. Anabitarte, M. Bellini, Phys. Lett. B **640**, 126 (2006)
24. M. Bellini, Phys. Lett. B **632**, 610 (2006)
25. M. Bellini, Phys. Lett. B **609**, 187 (2005)
26. J.E.M. Aguilar, M. Bellini, Phys. Lett. B **619**, 208 (2005)
27. J. Ponce de Leon, Gen. Relat. Grav. **20**, 539 (1988)
28. T.S. Bunch, P.C.W. Davies, Proc. R. Soc. London A **360**, 117 (1978)
29. A. Raya, J.E.M. Aguilar, M. Bellini, Phys. Lett. B **638**, 314 (2006)
30. Particle Data Group, Phys. Lett. B **592**, 206 (2004)